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Generalized Negative Binomial Distributions

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We show that a wide variety of processes lead, in certain limit, to a simple generalization of the negative binomial distribution. Its properties are studied in detail and used to derive an important result in the theory of avalanches.

KEY WORDS: Point processes; clusters; galaxy formation.

1. INTRODUCTION

In this work we shall consider distributions of points generated by certain type of processes which are relevant to a variety of physically interesting situations.

The quantity we shall be interested in is the probability, $P_n(V)$, that a volume, V, placed at random in the resulting distribution contains n points. Using some techniques we manage to obtain this quantity in terms of the properties of the processes generating the distribution and find that in an appropriate limit $P_n(V)$ reduces to the negative binomial distribution (NBD). This distribution have been found (Carruther and Shih, 1983; Carruther and Minh, 1983) to be a good approximation to the distribution of Zwicky clusters and it also arises in the statistics of hadron multiplicities in high-energy particle collisions. So, it is interesting to see that this distribution appears as a limit case of distributions generated by rather common processes. We shall also be able to determine that those distributions of points whose $P_n(V)$ is a NBD, are made up of randomly distributed clusters and determine their multiplicity function.

We shall find that the cluster multiplicity function may be expressed as a superposition of avalanches of a certain type starting at different times and use this fact to obtain the corresponding avalanche size distribution.

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We shall first deal with the case of linear avalanches, corresponding to the NBD, and then extend the procedure to more general cases.

Now, before giving the precise mathematical characterization of the processes that we shall consider, we discuss an example: seeds of some plant driven by the wind have an uniform probability for ending up at any position within a certain area. When a seed falls at some place it originates a plant that, in turn, will produce new seeds. So the probability per unit of time for a new seed to fall in the neighbourhood of a previously fallen one is greatly enhanced with respect to that for a randomly chosen point, where only seeds carried by the wind (having originated from plants outside the area in question) can fall. In general the processes under consideration include situations where points have an uniform probability for appearing at any position far from previously placed points, but each point becomes also a generator of new points enhancing this probability in their neighbourhood.

To give the precise definition of the processes to be studied, we assume that the points are distributed in three dimensional space, although the dimensionality of the underlying space is irrelevant to most of our considerations. Take a region of space with volume Ω devoid of points and where the probability per unit of time and volume for a point to appear is certain constant L, so that the first point is equally likely to appear at any position within Ω. The probability per unit of time and volume for a second point to appear at position $\vec{x}$ is $L(1+w(r))$ where $w(r)$ is some function of the distance, r, from the first point to $\vec{x}$. In general, the probability per unit of time and volume for the ith point to appear at position $\vec{x}$ is:

$$L\left(1+\sum_{j=1}^{i-1} w(|\vec{x}-\vec{x}_j|)\right) \tag{1}$$

where $\vec{x}_j$ is the position of the jth point. The physical content of expression (1) is that the enhancements of the probability density due to the presence of various points add linearly.

Notice that although for convenience we have considered L to be a constant in time, we could have chosen for it any function of time. In fact, it is clear that the properties of the distribution at a time when there are N points within Ω do not depend on how fast the distribution have been created, but on the probabilities for the ith point to be placed at different positions within Ω.

2. DERIVATION OF $P_n(V)$

The function $w(r)$ completely defines the processes under consideration. So we could in principle obtain the correlation functions corresponding to the resulting distribution and express $P_n(V)$ by their means (White, 1979). This would be, however, a very inconvenient approach, since there is not a straightforward procedure for obtaining the correlation functions from $w(r)$. The advantage of the present approach lies on the fact that $P_n(V)$ may be obtained directly in terms of $w(r)$, bypassing the computation of the correlation functions. This approach is interesting in itself, since it may be useful to deal with distributions generated by processes others than those considered here.

It is important to realize that although the two point (and higher order) correlation function may be obtained from $w(r)$, no simple identification between these two functions is posible, since their meanings are substantialy different. The two point correlation function, $\xi(r)$, gives the fractional enhancement of the probability density for finding a point at a distance r from a point chosen at random in the resulting distribution, that is, at the time when there are N points within Ω. The definition of $\xi(r)$ does not take into account the order in which the point have been placed, so it does not distinguish between cause and effect. This distinction is taken into account, however, in the definition of $w(r)$, since it represents the fractional enhancement of the probability density at a position at a distance r from a previously placed point due to the presence of this point, which is the cause of that enhancement. At the stage when the jth point is about to be placed, we only need to count the enhancements due to the $j-1$ previously placed points, since in any actual physical process, the probability density for the jth point can not depend on the position of points still to be placed.

To obtain $P_n(V)$, we shall first obtain $P_0(V)$ and use the following relationship (Otto *et al.* 1986)

$$P_n(V) = \frac{(-\bar{n})^n}{n!} \frac{d^n}{d\bar{n}^n} P_0(V) \tag{2}$$

where $\bar{n}$ is the mean density of points and where the derivatives with respect to $\bar{n}$ are carried out holding the correlations fixed. This is the same as deriving with respect to $\bar{n}$ holding $w(r)$ fixed, since the correlations functions depends only on $w(r)$. It is this last fact that imply the validity of expression (2) in the present problem.

We shall first obtain $P_n(V)$ in the cases when:

$$\left| w - \frac{1}{V} \int_V \int_V w(|\vec{x}_1 - \vec{x}_2|)\, d\vec{x}_1\, d\vec{x}_2 \right| \ll w$$
$$w \equiv 4\pi \int_0^\infty w(r)\, r^2\, dr \tag{3}$$

That is, in the cases where $w(r)$ converges on scales much smaller than V. The result that we shall obtain is strictly valid in the limit in which this scale goes to zero.

Expression (1) implies that the probability density, $\tau(\vec{x}, i)$ for the ith point to be placed at $\vec{x}$ is given by

$$\tau(\vec{x}, i) = \frac{1 + \sum_{j=1}^{i-1} w(|\vec{x} - \vec{x}_j|)}{\int_\Omega (1 + \sum_{j=1}^{i-1} w(|\vec{x} - \vec{x}_j|)) \, d\vec{x}} \tag{4}$$

Using the definition in (3), we have

$$\int_\Omega \left(1 + \sum_{j=1}^{i-1} w(|\vec{x} - \vec{x}_j|)\right) d\vec{x} = \Omega + (i-1)\, w$$

By means of $\tau(\vec{x}, i)$, we may obtain the probability, $P_0(V)$, of not finding any point within a randomly placed volume V when there are N points within Ω ($\Omega \gg V$).

$$P_0(V) = \Pi_{i=1}^{N} \left(1 - \int_V \tau(\vec{x}, i)\, d\vec{x}\right) \tag{5}$$

where the ith factor is the probability for the ith point to be placed outside V. When the ith point is placed in the distribution there are, ex hypothesis, no points within V, so the integral over V of the terms of the form $w(|\vec{x} - \vec{x}_j|)$ in (4) must be equal to zero in the limit in which $w(r)$ goes to a delta distribution. We then have:

$$\begin{gathered} \int_V \tau(\vec{x}, i)\, d\vec{x} = \frac{V}{\Omega + (i-1)\, w} \\ P_0(V) = \Pi_{i=1}^{N} \left(1 - \frac{V}{\Omega + (i-1)\, w}\right) \end{gathered} \tag{6}$$

In the large Ω, N limit, which is the interesting case we have:

$$\begin{gathered} P_0(V) = \exp\left[-\int_0^{\bar{n}} \frac{V\, dx}{1 + xw}\right] = (1 + w\bar{n})^{-V/w} \\ \bar{n} = \lim_{\Omega \to \infty} \frac{N}{\Omega} \end{gathered} \tag{7}$$

Using expression (2)

$$P_n(V) = \frac{(-\bar{n})^n}{n!} \frac{d^n}{d\bar{n}^n} (1+w\bar{n})^{-V/w} = \frac{\Gamma(V/w+n)}{\Gamma(V/w)} \frac{1}{n!} \frac{(\bar{n}w)^n}{(1+w\bar{n})^{V/w+n}}$$
$$= \frac{(\bar{n})^n}{n!} [\Pi_{i=0}^{n-1}(V+\omega i)](1+w\bar{n})^{-(V/w+n)} \tag{8}$$

where the derivatives are carried out holding w fixed. Γ stands for the gamma function. This is formally equal to the expression for the NBD, but here the parameter V/w is not restricted to take integer values.

Using expression (2), it is easy to show that the generating function, $G[P_n(V)](t)$, of $P_n(V)$ is related to $P_0(V, \bar{n})$ by:

$$G[P_n(V)](t) = \sum_{n=0}^{\infty} P_n(V)\, e^{nt} = P_0(V, (1-e^t)\,\bar{n}) \tag{9}$$

So, in the present case we have

$$G[P_n(V)](t) = (1 + \bar{n}w(1-e^t))^{-V/w} \tag{10}$$

We then have for the first two moments of $P_n(V)$

$$\langle n \rangle = \bar{n}V; \qquad \langle (n - \langle n \rangle)^2 \rangle = \bar{n}V(1+\bar{n}w) \tag{11}$$

So we find the following relationship between w and the two point correlation function, ξ

$$w = \frac{1}{V} \int_V \int_V \xi(|\vec{x}_1 - \vec{x}_2|)\, d\vec{x}_1\, d\vec{x}_2 \tag{12}$$

To obtain $P_0(V)$ we have used the first expression in (6). This expression holds in the limit of arbitrarily small scale of convergence of $w(r)$. In this limit the enhancement of the probability density within V due to points outside it becomes negligible. We shall now consider a situation where the scale of $w(r)$ is finite so that expression (6) is no longer valid. To compute the enhancement of the probability density within V due to points outside it, we assume that these points are randomly distributed. Then expression (6) takes the form:

$$P_0(V) = \Pi_{i=1}^{N} \left(1 - \frac{V + (i-1)\,\Delta}{\Omega + (i-1)\,w}\right);$$
$$\Delta = \frac{1}{\Omega} \int_\Omega \int_\Omega w(|\vec{x}_1 - \vec{x}_2|)\, C(\vec{x}_2)(1 - C(\vec{x}_1))\, d\vec{x}_1\, d\vec{x}_2 \tag{13}$$

where $C(\vec{x})$ is a function that is equal to one for points within V and equal to zero otherwise. Taking, as before, N, Ω to infinity, we find

$$P_0(V) = e^{-q\bar{n}/w}(1+w\bar{n})^{-V/w+q/w^2}$$
$$q \equiv \Omega\Delta \tag{14}$$

Since the correlations remain fixed when $w(r)$ is held fixed, we may obtain $P_n(r)$ by deriving this expression with respect to $\bar{n}$ holding p, q, V fixed as indicated in (2).

Expression (14) is only an approximation, since in obtaining the mean value of the enhancement of the probability within V due to outside points we have neglected the correlations. This leads, however, to a small error.

3. STRUCTURE OF THE DISTRIBUTION ASSOCIATED WITH THE NBD

We shall now discuss the spatial structures of those distributions of points whose $P_n(V)$ is a generalized NBD.

To obtain expression (8) we have assumed that $w(r)$ is a delta distribution. This implies that when a new point is placed in the distribution there is some probability that it falls on top of a previously placed point and that the probability density for it to be placed at any position where there are not points is uniform. Hence, the resulting distribution is made up of randomly distributed point-like clusters characterized by certain multiplicity function, $P(N)$.

We shall now describe the general procedure whereby it may be stablished whether a given $P_n(r)$ corresponds to a distribution of points containing point-like clusters and shall determine their multiplicity function. To this end we use the fact that in the small V limit, $P_n(V)$ is approximately proportional to the probability for a cluster to contain n points. This is so because the probability for V to contain more than one cluster becomes negligible in this limit. More precisely, we may write

$$\lim_{V\to 0} \frac{(\bar{n}V/\langle N\rangle)\, P(N=n)}{P_n(V)} = 1 \tag{15}$$

where $\langle N \rangle$ is the mean number of points in a cluster, so that $\bar{n}/\langle N \rangle$ is the mean density of clusters and the parentheses in the numerator is the probability for the volume V to contain one cluster. We may then write:

$$P(N=n)=\langle N \rangle \lim_{V \to 0} \frac{P_n(V)}{\bar{n}V} \equiv \langle N \rangle F(n)$$
$$P(N)=\frac{F(n=N)}{\sum_{n=1}^{\infty} F(n)} \tag{16}$$

If, unlike in the present case, the correlation functions of the point distribution corresponding to $P_n(V)$ were regular, so that the probability of having two or more points of the distribution at the same spatial point (position) was zero, it could not contain point-like clusters, and $P(N)$ must be equal to zero for N larger than one.

Using expression (8) in (16) we find in the present case:

$$F(n)=\frac{1}{\bar{n}w}\frac{1}{n} A^n; \qquad A \equiv \frac{\bar{n}w}{1+\bar{n}w}$$
$$P(N)=(-\ln(1-A))^{-1} \frac{1}{N} A^N \tag{17}$$

If a distribution of points is made up of randomly placed point-like clusters, the generating function, $G[P_n(V)](t)$, of the corresponding $P_n(V)$ and the generating function, $G[P(N)](t)$, of the cluster multiplicity function are related by:

$$G[P(N)](t)=1+\frac{\langle N \rangle}{\bar{n}V} \ln[G[P_n(V)](t)] \tag{18}$$

This expression may inmedeately be obtained by writting the generating function of $P_n(V)$ as the sum over all posible cases of the product of the Poissonian probability for volume V to contain j cluster times the generating function of $P_n(V)$ when this condition holds, which is equal to the jth power of the cluster generating function

$$G[P_n(V)]=e^{-\bar{n}V/\langle N \rangle} \sum_{j=0}^{\infty} \frac{(\bar{n}V/\langle N \rangle)^j}{j!} [G[P(N)](t)]^j \tag{19}$$

Thus, the necessary and suffcient condition for the clusters to be uncorrelated (randomly placed) is that the logarithm of the generating function of $P_n(V)$ be proportional to V, so that $P(N)$ be independent of V.

In our problem it is easy to check that this relationship is satisfied, since expession (17) gives:

$$G[P(N)](t) = \frac{\ln(1 - Ae^t)}{\ln(1 - A)} \tag{20}$$

and the generating function of $P_n(V)$ is given by (10).

It is interesting to note that, in principle, different point distributions may exist such that their corresponding $P_n(V)$ is a NBD. That is, we may generate distributions made up of correlated point-like clusters leading to this $P_n(V)$. Take the case of clusters with an exponential multiplicity function placed accordingly to a regular grid so that a certain volume V always contain the same number of clusters. It is clear that the corresponding $P_n(V)$ is a NBD, but this is so only for a particular volume (and shape). This is also true for any other distribution with or without point-like clusters. It is only for the distribution that we have described that $P_n(V)$ is a NBD for any volume.

In the process of generating the point distribution, new clusters are created whenever a new point is placed at a position where there are not other points. These points we term father points since they determine where a cluster will form. Once a father point is placed, the growth of the cluster is a typical avalanche process in which the probability per unit of time for a new point to appear is at any time given by the product of certain function of time by the number of points already present at that time.

If we knew the probability distribution for the sizes of these avalanches, we could express $P(N)$ (see (17)) as a combinations of distributions corresponding to avalanches starting at different times. This is so because $P(N)$ is the distribution corresponding to all clusters present at some given time, whose father points have been placed at different times. Here we shall solve the opposite problem, we shall use the expression that we have obtained for $P(N)$ to obtain the probability distribution for the sizes of avalanches starting at some given time. To this end we must first obtain the distribution in time of the father points. Instead of time itself, we shall use as a time variable the value of the mean density, $\bar{n}(t)$, which takes the value zero at the initial time and the value $\bar{n}$ at the final time when all points have been placed in the distribution.

From expression (4) it is clear that the probability for the point placed at time $\bar{n}(t)$ to fall outside any existing cluster, that is, the probability that it is a father point, is given by:

$$\frac{1}{1 + w\bar{n}(t)} \tag{21}$$

so, the probability, $P(\bar{n}(t))\, d\bar{n}(t)$, that a randomly chosen father point have been placed between $\bar{n}(t)$ and $\bar{n}(t)+d\bar{n}(t)$ is

$$P(\bar{n}(t))\, d\bar{n}(t)=\left(\int_0^{\bar{n}} \frac{d\bar{n}(t)}{1+\bar{n}(t)\, w}\right)^{-1} \frac{d\bar{n}(t)}{1+\bar{n}(t)\, w}$$
$$=\frac{w}{\ln(1+w\bar{n})}\frac{d\bar{n}(t)}{1+\bar{n}(t)\, w} \tag{22}$$

We may then write for $P(N)$

$$P(N)=\int_0^{\bar{n}} P(\bar{n}(t))\, P(N/\bar{n}(t))\, d\bar{n}(t)=\frac{1}{\ln(1+w\bar{n})}\frac{1}{N}\left(\frac{\bar{n}w}{1+\bar{n}w}\right)^N \tag{23}$$

where $P(N/\bar{n}(t))$ is the probability distribution for the sizes at time $\bar{n}$ of clusters whose father points have been placed at time $\bar{n}(t)$. The quantities $\bar{n}$, $\bar{n}(t)$ may enter $P(N/\bar{n}(t))$ only through the variable:

$$u\equiv\int_{\bar{n}(t)}^{\bar{n}} \frac{dx}{1+wx}=\frac{1}{w}\ln\frac{1+\bar{n}w}{1+\bar{n}(t)\, w}$$

which reflects the fact that the size, at time $\bar{n}$, of a cluster initiated at time $\bar{n}(t)$ depends on the number of points still to be placed in the distribution and that the probability for a point to be placed in a cluster initiated at $\bar{n}(t)$ is reduced by the factor $(1+w\bar{n}(t))^{-1}$ with respect to the corresponding probability for the first cluster, initiated at $\bar{n}(t)=0$. This fact, together with expression (23) completely determine $P(N/\bar{n}(t))$:

$$P(N/\bar{n}(t))=e^{-wu}(1-e^{-wu})^{N-1}=\frac{1+\bar{n}(t)\, w}{1+\bar{n}w}\left(\frac{(\bar{n}-\bar{n}(t))\, w}{1+\bar{n}w}\right)^{N-1} \tag{24}$$

So, for the cluster initiated at $\bar{n}(t)=0$, we have:

$$P(N/0)=\frac{1}{1+\bar{n}w}\left(\frac{\bar{n}w}{1+\bar{n}w}\right)^{N-1} \tag{25}$$

We may then conclude that the avalanche process corresponding to the growth of these clusters leads to an exponential distribution. The same conclusion applies for any avalanche process in which the probability per unit time, Z, for a new point to joint the avalanche may be written in the form:

$$Z(t)=L(t)\, n(t) \tag{26}$$

where $L(t)$ is some function of time and $n(t)$ is the number of points in the avalanche at time t. In this case the size of the avalanche at time t is given by expression (25) with $\bar{n}w$ given by:

$$\bar{n}w = \exp\left[\int_0^t L(t)\,dt\right] - 1 \tag{27}$$

When $L(t)$ is a constant, the expected value of the size of the avalanche growes exponentialy with time.

We will now describe the generalization of this procedure to non-linear avalanches, where the probability per unit time for a new point to join the avalanche, $Z(t)$, may be expressed in the form:

$$Z(t) = L(t)\ f(n(t))$$

f being a non-linear function of $n(t)$.

To this end, we consider processes whereby points are sequentialy placed within a given volume and where the enhancement of the probability density due to the previously placed points add up non-linearly, unlike in the case leading to expression (7). More precisely, the probability density for the placement of the ith point at position $\vec{x}$ is proportional to:

$$\left(1 + g\left(\sum_{j=1}^{i-1} W(|\vec{x} - \vec{x}_j|\right)\right) \tag{28}$$

where $g(u)$ is some non-linear function of its argument and $w(r)$ is defined as in expression (1).

In the limit in which $w(r)$ goes to a delta distribution, which is the one relevant to the theory of avalanches, the distribution of points resulting from the above process is made up of randomly placed point-like clusters. However, even in this limit, for a non-linear $g(u)$, the processes described by (28) are not clearly related to the avalanche processes. The reason being that when a point is placed within the probability enhancement range of a previously placed one, their regions of probability enhancement do not exactly overlap. This implies that a proper treatment of processes (28) must involve complicated geometrical coefficients bearing no relationship with the avalanches under consideration. To avoid this problem we shall consider a slightly different kind of processes whereby after depositing a point according to (28), when it happens to fall within the range of a previously placed one, the former is moved to the position of the latter so that both ranges exactly overlap.

Note that in the relevant zero range limit there is no difference between (28) and the newly defined processes in regard to the position of the points. But the difference between the resulting multiplicity function persist in this limit, leading obviously to higher values (if $\omega(r) > 0$) in the last case, which is the one that makes posible an exact treatment of the avalanches. So, these are the processes considered in what follows.

When $w(r)$ is a δ distribution, only points within the same cluster (at $\vec{x}$) contribute to the sum in (28), and it is easy to realize that instead of expression (6) we should have for the probability that the ith point is placed within volume V

$$\frac{V}{\Omega + (i-1)(\bar{g}((i-1)/\Omega))}$$
$$\bar{g}(\bar{n}_i) \equiv \frac{1}{\langle N \rangle} \sum_{N=1}^{\infty} g(\omega N)\, P(N, \bar{n}_i); \qquad \bar{n}_i \equiv \frac{i-1}{\Omega} \tag{29}$$

$P(N, \bar{n}_i)$ is the cluster size distribution when the mean density of points is $\bar{n}_i$. The reason for this expression is that the sum in (28), within a cluster containing N points, is equal to ω times N.

Following the same procedure that led us to expression (7) we find

$$P_0(V, \bar{n}) = \exp\left[-\int_0^{\bar{n}} \frac{V\, dx}{1 + x\bar{g}(x)} \right] \tag{30}$$

From this expression and using expression (9) and (18) we may obtain the clusters multiplicity function, $P(N, \bar{n})$, that may be used to compute $\bar{g}(\bar{n})$. Now, since this function enters $P(N, \bar{n})$ through (30), we must solve for it selfconsistently. If $g(u)$ is of the form

$$g(u) = u + \alpha u^2 + \beta u^3 + \cdots$$

we have

$$\bar{g}(x) = \omega + \alpha f_1(x) + \beta f_2(x) + \cdots$$

where

$$f_j(x) = \frac{\langle N^{j+1} \rangle}{\langle N \rangle}$$

These averages are computed using $P(N, x)$. That is, they correspond to the cluster presents when $\bar{n} = x$.

As an example we will determine f_1, f_2 in the case when $g(u)$ contains up to the cubic term. Then, using expressions (30), (9), (18), we find that the generating function of $P(N, \bar{n})$ is given by:

$$G(P(N,\bar{n}))(t) = 1 - \frac{\langle N \rangle}{\bar{n}V} \int_0^{\bar{n}(1-e^t)} \frac{V\,dx}{1 + x(w + \alpha f_1(x) + \beta f_2(x))} \tag{31}$$

We then have

$$\begin{gathered} \langle N \rangle = G'(t=0); \quad \langle N^2 \rangle = G''(t=0) = \langle N \rangle\, f_1(\bar{n}) \\ \langle N^3 \rangle = G'''(t=0) = \langle N \rangle\, f_2(\bar{n}) \end{gathered} \tag{32}$$

These equations imply that

$$\begin{aligned} f_1(\bar{n}) &= 1 + \bar{n}(w + \alpha + \beta) \\ f_2(\bar{n}) &= 1 + 3\bar{n}(w + \alpha + \beta) + 2\bar{n}^2 w(w + \alpha + \beta) \end{aligned} \tag{33}$$

Substituting these expressions in (31) we may finally obtain the generating function of $P(N, \bar{n})$. We then have

$$\begin{aligned} P(N,\bar{n}) &= \left.\frac{d^n G(P(N,\bar{n}))}{d(e^t)^n}\right|_{e^t=0} = \frac{\langle N \rangle(-\bar{n})^{n-1}}{n!} \frac{d^n Y(\bar{n})}{d\bar{n}^n} \\ Y(\bar{n}) &\equiv \int_0^{\bar{n}} \frac{dx}{1 + x(w + \alpha f_1(x) + \beta f_2(x))} \end{aligned} \tag{34}$$

In the case when $\beta = 0$ and $w > 3\alpha$ we have explicitly

$$Y(\bar{n}) = \frac{1}{\sqrt{(w+\alpha)(w-3\alpha)}} \left[\ln \frac{2(w+\alpha)\,\alpha x + w + \alpha - \sqrt{(w+\alpha)(w-3\alpha)}}{2(w+\alpha)\,\alpha x + w + \alpha + \sqrt{(w+\alpha)(w-3\alpha)}} \right]_0^{\bar{n}} \tag{35}$$

Using this in (34) and normalizing to obtain $\langle N \rangle$, we find

$$\begin{aligned} P(N,\bar{n}) &= \frac{1}{N} \left(\ln \frac{1 - g/(g+b)}{1 - g/(g+a)} \right)^{-1} \left[\left(\frac{g}{g+a}\right)^N - \left(\frac{g}{g+b}\right)^N \right] \\ g &= 2\left(1 + \frac{\alpha}{w}\right) \frac{\alpha}{w}\, \bar{n} w \\ a &= 1 + \frac{\alpha}{w} - \sqrt{(1 - 3\alpha w)\left(1 + \frac{\alpha}{w}\right)} \\ b &= 1 + \frac{\alpha}{w} + \sqrt{(1 - 3\alpha w)\left(1 + \frac{\alpha}{w}\right)} \end{aligned} \tag{36}$$

This distribution corresponds to a mixture of avalanches starting at different times. These avalanches are such that the probability per unit time, $Z(t)$, for a new point to joint the avalanche is given by

$$Z(t) = L(t)\left(n(t) + \frac{\alpha}{w}\, n^2(t)\right) \tag{37}$$

where $n(t)$ stands for the number of points present at time t, and $L(t)$ is a certain function of time that determines how fast the avalanche grows, but not the size distribution at a given time (characterized by $\langle N \rangle$).

To obtain the distribution corresponding to an avalanche starting at time $\bar{n}(t)$, $P(N/\bar{n}(t))$, we must solve an equation similar to (23). In the present case the right hand side of the equation is given by expression (36); the time distribution for the father points, $P(\bar{n}(t))$, may be obtained after the same considerations that led to expression (22)

$$\begin{aligned} P(\bar{n}(t))\, d\bar{n}(t) &= \left(\int_0^{\bar{n}} \frac{dx}{1 + wx + \alpha x(1 + x(w+\alpha))}\right)^{-1} \\ &\quad \times \frac{d\bar{n}(t)}{1 + \bar{n}(t)\, w + \alpha \bar{n}(t)(1 + \bar{n}(t)(w+\alpha))} \\ &\equiv (Y(\bar{n}))^{-1}\, dY(\bar{n}(t)) \end{aligned} \tag{38}$$

and $P(N/\bar{n}(t))$ can only depend on the variable

$$u \equiv \int_{\bar{n}(t)}^{\bar{n}} \frac{dx}{1 + xw + \alpha x(1 + x(w+\alpha))} = Y(\bar{n}) - Y(\bar{n}(t))$$

We may then write the present case equivalent of expression (23) in the form:

$$(Y(\bar{n}))^{-1} \int_0^{Y(\bar{n})} P_1(N/u)\, du = P(N, \bar{n}) \tag{39}$$

where $P_1(N/u)$ is simply $P(N/\bar{n}(t))$ expressed as a function of u and $P(N, \bar{n})$ is given by (36). We have used the fact that $du = -dY$. By deriving this expression with respect to $\bar{n}$ we find

$$P(N/0) = P_1(N/Y(\bar{n})) = P(N, \bar{n}) + \frac{Y(\bar{n})}{Y'(\bar{n})} \frac{d}{d\bar{n}} P(N, \bar{n}) \tag{40}$$

The prime stands for derivative with respect to $\bar{n}$. $P(N/0)$ is the size probability distribution at time $\bar{n}$ of clusters initiated at $\bar{n}(t)=0$. For clusters initiated at $\bar{n}(t)$ we have inmedeatly

$$\begin{aligned} P(N/\bar{n}(t)) &\equiv P_1(N/u) \equiv P_1(N/Y(\bar{n}_1)) \\ u &= Y(\bar{n}) - Y(\bar{n}(t)) \equiv Y(\bar{n}_1) \end{aligned} \tag{41}$$

This equation defines $\bar{n}_1$ that when placed in the right hand side of (40) gives the result searched for. However, we do not need this procedure, since all these avalanches belong to type (37) only that with a different $L(t)$ and starting time. So the size distribution as a function of $\langle N \rangle$, α/w is the same for all of them and we may obtain it using simply $P(N/0)$.

Expression (40) is valid in general, that is, for any $\bar{g}(x)$, with $Y(\bar{n})$ defined by:

$$Y(\bar{n}) = \int_0^{\bar{n}} \frac{dx}{1 + x\bar{g}(x)} \tag{42}$$

and $P(N, \bar{n})$ given by (34). However, only when $\bar{g}(x)$ is a polinomia can we find, through arguments similar to those leading to (33), its relationship with $g(u)$ which is directly related to the definition of the avalanche process.

Obtaining the size probability distribution for an avalanche $Z(t) = L(t)\ f(\bar{n}(t))$ with f a polinomia is a straightforward exercise leading always to expressions made up of a finite number of simple analytical functions. To this end we must use $g(u) = f(\bar{n}(t) = u)$, obtain from it $\bar{g}(u)$ (which is somewhat tedious for high order polinomia) use it in (42) to obtain $Y(\bar{n})$, which give us $P(N, \bar{n})$ through (34), and substitute for this two quantities in (40).

In the case that we are presently considering, substituting (36) in (40) we find

$$\begin{aligned} P(N/0) = {} & 2\left(1 + \frac{\alpha}{w}\right)\frac{\alpha}{w}\left(1 + \bar{n}w + \frac{\alpha}{w}\,\bar{n}w\left(1 + \bar{n}w\left(1 + \frac{\alpha}{w}\right)\right)\right) \\ & \times \left[\frac{1}{\sqrt{(1+\alpha/w)(1-3\alpha/w)}}\left(\left(\frac{g}{g+a}\right)^{N-1}\frac{a}{(g+a)^2}\right.\right. \\ & \left.\left. - \left(\frac{g}{g+b}\right)^{N-1}\frac{b}{(g+b)^2}\right) - \frac{2P(N,\bar{n})}{(g+a)(g+b)}\right] + P(N,\bar{n}) \end{aligned} \tag{43}$$

This distribution corresponds to a particular avalanche of type (37), namely, that taking place in the first cluster created by a certain type of processes leading to a point distribution with mean density $\bar{n}$. To obtain the distribution for any avalanche of type (37), where $\bar{n}$ have no meaning, we must express $\bar{n}w$ in terms of $\langle N \rangle$. However, since the expression is not handy it is better to use (43) with $\bar{n}w$ as a parameter.

So far, we have not considered the time evolution of the avalanches, caring only about the shape of the distribution for a given value of $\langle N \rangle$. However, it is easy to obtain, for an avalanche defined by $Z(t) = L(t)\, f(n(t))$, $\langle N \rangle$ as a function of time. To this end we only need to choose the deposition rate, given by $\bar{n}(t)$, so that the points are placed within the first cluster at a rate given by $Z(t)$. Using (38) it may be shown that one should have

$$w\,Y(\bar{n}(t)) = \int_0^t L(t)\,dt \tag{44}$$

From this expression we obtain $\bar{n}(t)$ that may be used in (40) in the place of $\bar{n}$ to calculate $\langle N \rangle(t)$.

In the present case we find:

$$\begin{aligned} w\bar{n}(t) &= \left(1 + \frac{\alpha}{w}\right)^{-1} \frac{a}{\alpha/w} \frac{A-1}{1-(a/b)\,A} \\ A &= \exp\left[\sqrt{\left(1+\frac{\alpha}{w}\right)\left(1-\frac{3\alpha}{w}\right)} \int_0^t L(t)\,dt\right] \end{aligned} \tag{45}$$

$\langle N \rangle(t)$ may be obtained inmedeately by means of (43).

4. CONCLUDING REMARKS

We have seen that the generalized binomial is a limit of distributions resulting from a very general type of processes. This distribution is made up of randomly distributed point-like clusters whose multiplicity function we have obtained. These results have been obtained in the limit of zero range for the enhancement of probability. When this range is finite, we have seen (expression 14) that $P_n(V)$ changes. However, the cluster multiplicity function remain unchanged as long as the clusters do not percolate; that is, as long as no cluster contains more than a father point.

Considering the evolution of any individual cluster through the deposition process we have realized that they grow like an avalanche of a certain type. So, the cluster multiplicity function may be expressed as as superposition of avalanches of the same time starting at different times following a well defined time pattern. From this fact we have infered that the size of avalanches starting all at the same type follow an exponential distribution. The avalanches implied in these considerations are linear (the rate of growth being proportional to size). These type of avalanches is rather common, thus the distribution mentioned above have considerable interest.

Furthermore, by considering processes with non-linear enhancement of the probability density, we have shown how to obtain the probability distribution for the size of non-linear avalanches, and how the mean size of the avalanche grows with time. This we have done in the case that $f(n(t))$ is a polinomia. in general case the procedure described here also applies, but the relationship between $g(u)$ and $\bar{g}(x)$ must be obtained by other means.

We have only considered sequential avalanches, where points joint to it one by one. However, it is straightforward to generalize the procedure to the case when the points may come in clumps, provided that their multiplicity function does not depend on $n(t)$.

In the processes we have considered so far, the father points were uniformly distributed. This restriction, however, may be removed quite easily. The position of the father points may be the result of a non-uniform Poissonian process with probability density $\rho(\vec{x})$

$$P_0(V) = \int_0^{\infty} [(1 + w\bar{n})^{-V/w}]^{(1+\delta)} P(\delta)\, d\delta$$
$$\delta \equiv \frac{M - \langle M \rangle}{\langle M \rangle}; \qquad M \equiv \int_V \rho(\vec{x})\, d^3\vec{x} \tag{46}$$

where M is the integral within a randomly placed volume V of the probability density for the father points and $P(\delta)$ is its probability distribution. $P_n(V)$ may be obtained using expression (2) and the generating function may be obtained by substituting in $P_0(V)$ $\bar{n}$ by $\bar{n}(1 - e^t)$. Using this result we find for the second and third central moments of $P_n(V)$ in terms of the moments of the underlying probability density field:

$$\langle (n - \langle n \rangle)^2 \rangle = V\bar{n}(1 + w\bar{n} + \bar{n}V\langle \delta^2 \rangle)$$
$$\langle (n - \langle n \rangle)^3 \rangle = \langle \delta^3 \rangle + 3(1 + w\bar{n})\frac{\langle \delta^2 \rangle}{\bar{n}V} + \frac{1 + 2(w\bar{n})^2 + 3w\bar{n}}{(\bar{n}V)^2} \tag{47}$$

In this case the distribution also contains point-like clusters, but they are now correlated due to the correlations of the probability density field.

If in expression (46) we change expression (7) (the bracket) by expression (14) the resulting distribution corresponds to a very large variety of processes where the underlying probability density is not uniform and the enhancement of this density due tho the presence of a point extends to some finite distance from it.

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